



## On Singular Values of Hankel Operators of Finite Rank\*

W. B. Gragg

*Department of Mathematics  
Naval Postgraduate School  
Monterey, California 93943*

and

L. Reichel

*Bergen Scientific Centre  
Allégaten 36 Universitetet  
N-5007 Bergen, Norway  
and  
Department of Mathematics  
University of Kentucky  
Lexington, Kentucky 40506*

Submitted by Michael Neumann

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### ABSTRACT

Let  $H$  be a Hankel operator defined by its symbol  $\rho = \pi/\chi$ , where  $\chi$  is a monic polynomial of degree  $n$  and  $\pi$  is a polynomial of degree less than  $n$ . Then  $H$  has rank  $n$ . We derive a generalized Takagi singular value problem defined by two  $n \times n$  matrices, such that its  $n$  generalized Takagi singular values are the positive singular values of  $H$ . If  $\rho$  is real, then the generalized Takagi singular value problem reduces to a generalized symmetric eigenvalue problem. The computations can be carried out so that the Lanczos method applied to the latter problem requires only  $O(n \log n)$  arithmetic operations for each iteration. If  $\pi$  and  $\chi$  are given in power form, then the elements of all  $n \times n$  matrices required can be determined in  $O(n^2)$  arithmetic operations.

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## 1. INTRODUCTION

Let  $H = [\eta_{j+k}]_{j,k=0}^{\infty}$  be a Hankel operator defined by its rational symbol  $\rho = \pi/\chi$ , where

$$\pi(\lambda) := \sum_{j=0}^{n-1} \pi_j \lambda^j \quad \text{and} \quad \chi(\lambda) := \sum_{j=0}^n \chi_j \lambda^j, \quad \chi_n = 1. \quad (1.1)$$

We assume that  $\pi$  and  $\chi$  have no common zeros. The elements  $\eta_j$  of  $H$  are then given by

$$\rho(\lambda) = \frac{\pi(\lambda)}{\chi(\lambda)} = \sum_{j=0}^{\infty} \eta_j \lambda^{-j-1}. \quad (1.2)$$

In order to simplify our presentation, we assume that the zeros  $\{\lambda_k\}_{k=1}^n$  of  $\chi$  are distinct. How our formulas need to be modified in order to remove this assumption is discussed in Remark 1.1 below. Hence  $\rho$  has a partial fraction decomposition

$$\rho(\lambda) =: \sum_{k=1}^n \frac{\alpha_k}{\lambda - \lambda_k}. \quad (1.3)$$

Expansion of the right hand side of (1.3) in a geometric series, and comparison with (1.2), yields

$$\eta_j = \sum_{k=1}^n \alpha_k \lambda_k^j. \quad (1.4)$$

We now express (1.4) in matrix form. Let

$$A := \text{diag}[\alpha_1, \alpha_2, \dots, \alpha_n] \in \mathbb{C}^{n \times n}, \quad (1.5)$$

$$\Lambda := \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n] \in \mathbb{C}^{n \times n}, \quad (1.6)$$

and introduce the Vandermonde matrix

$$V_0 := [\lambda_{k+1}^j]_{j,k=0}^{n-1} \in \mathbb{C}^{n \times n}. \quad (1.7)$$

Define

$$V := [V_j]_{j=0}^{\infty} \in \mathbb{C}^{\infty \times n}, \quad (1.8)$$

where

$$V_j := V_0 \Lambda^j, \quad j \geq 1. \quad (1.9)$$

Then (1.4) can be written as

$$H = VAV^T. \quad (1.10)$$

Let  $l^2$  be the vector space of simply infinite column vectors with finite Euclidean norms.

**PROPOSITION 1.1.**  $H: l^2 \rightarrow l^2$  bounded  $\Leftrightarrow |\lambda_k| < 1$  for  $1 \leq k \leq n$ .

*Proof.* The proposition holds independent of the multiplicity of the  $\lambda_k$ . In the present proof we assume that the  $\lambda_k$  are distinct. The proof for confluent  $\lambda_k$  is commented on in Remark 1.1.

Let  $e_1 = [\varepsilon_j]_{j=0}^{\infty} \in \mathbb{C}^{\infty}$  be the axis vector with  $\varepsilon_0 = 1$ . Then

$$\begin{aligned} h = [\eta_j]_{j=0}^{\infty} &:= He_1 \in l^2 \Rightarrow \eta_j \rightarrow 0 \quad \text{as } j \rightarrow \infty \\ &\Rightarrow |\lambda_k| < 1 \quad \text{for } 1 \leq k \leq n, \end{aligned}$$

where the last implication follows from (1.4).

Conversely, assume that  $|\lambda_k| < 1$  for  $1 \leq k \leq n$ . Then by (1.8)–(1.10) we obtain

$$\begin{aligned} \|H\|_2 &\leq \|A\|_2 \|V\|_2^2 \leq \|A\|_2 \|V_0\|_2^2 \left\| \sum_{j=0}^{\infty} (\Lambda^H \Lambda)^{nj} \right\|_2^2 \\ &= \|A\|_2 \|V_0\|_2^2 \left\| \left[ I - (\Lambda^H \Lambda)^n \right]^{-1} \right\|_2^2. \end{aligned} \quad \blacksquare$$

We assume henceforth that  $|\lambda_k| < 1$  for  $1 \leq k \leq n$ . Introduce

$$U := VV_0^{-1} \quad (1.11)$$

$$H_0 := V_0 A V_0^T. \quad (1.12)$$

Then  $H_0$  has rank  $n$ . We note, by comparing (1.12) with (1.10), that  $H_0$  is the leading principal  $n \times n$  submatrix of  $H$ . From (1.10)–(1.12) it follows that

$$H = UH_0U^T. \quad (1.13)$$

The leading  $n \times n$  submatrix of  $U$  is  $I_n$ , the  $n \times n$  identity matrix.  $U$  therefore is of rank  $n$  and can be factored

$$U = QR, \quad Q \in \mathbb{C}^{\infty \times n}, \quad R \in \mathbb{C}^{n \times n},$$

where  $Q^H Q = I_n$  and  $R$  is a nonsingular right triangular matrix. We obtain

$$\sigma_+(H) = \sigma_+(QRH_0R^TQ^T) = \sigma(RH_0R^T), \quad (1.14)$$

where  $\sigma$  denotes the set of singular values and  $\sigma_+$  denotes the subset of the positive ones.

The  $n \times n$  matrix  $RH_0R^T$  is complex symmetric. Takagi [10, 11] showed the existence of a complex symmetric singular value decomposition

$$RH_0R^T = W\Sigma W^T, \quad W \in \mathbb{C}^{n \times n}, \quad \Sigma = \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_n], \quad (1.15)$$

where  $W^H W = I_n$  and  $\sigma_j > 0$  are the singular values of  $RH_0R^T$ . In Section 2 we present an elementary proof of the existence of this decomposition. Let  $W = [w_1, w_2, \dots, w_n]$ ,  $w_j \in \mathbb{C}^n$ . Then (1.15) can be written as the Takagi singular value problem

$$RH_0R^T \bar{w}_j = w_j \sigma_j, \quad w_j^H w_k = \delta_{jk}, \quad 1 \leq j, k \leq n, \quad (1.16)$$

where the bar denotes complex conjugation and  $\delta_{jk}$  is Kronecker's  $\delta$  function. The problems (1.15)–(1.16) could be solved by the algorithm described in [2], but this would require  $RH_0R^T$  to be explicitly computed. In order to avoid

these matrix multiplications we let  $v_j := R^H w_j$  and obtain from (1.16) the *generalized Takagi singular value problem*

$$H_0 \bar{v}_j = (R^H R)^{-1} v_j \sigma_j, \quad v_j^H (R^H R)^{-1} v_k = \delta_{jk}, \quad 1 \leq j, k \leq n. \quad (1.17)$$

The solution of (1.17) requires  $(R^H R)^{-1}$  to be known. In Section 3 we show that

$$(R^H R)^{-1} = I - B_0 B_0^H, \quad (1.18)$$

where  $B_0 \in \mathbb{C}^{n \times n}$  is a triangular Toeplitz matrix. The elements of  $B_0$  and  $H_0$  can be determined from the coefficients of  $\pi$  and  $\chi$  in  $O(n \log n)$  arithmetic operations by the fast Fourier transform (FFT) method. This is demonstrated in Section 4. Section 5 shows that

$$R^H R = \overline{T_1 M_0 T_1^H}, \quad T_1, M_0 \in \mathbb{C}^{n \times n}, \quad (1.19)$$

where  $T_1$  and  $M_0$  are Toeplitz matrices, and describes a numerical scheme for the computation of this factorization from (1.16) in  $O(n^2)$  arithmetic operations. We also present a Hermitian factorization of  $R^H R$  into  $n \times n$  triangular matrices.

The factorization (1.19) may be of interest for the numerical solution of (1.17). Assume that the coefficients of  $\pi$  and  $\chi$  are real valued. Then  $H_0, (R^H R)^{-1} \in \mathbb{R}^{n \times n}$ , and (1.17) reduces to a generalized symmetric eigenvalue problem. The Lanczos method [9, Section 15.11; 3] would appear suitable for solving this eigenproblem for the following reason. Let  $C \in \mathbb{C}^{n \times n}$  be a Hankel or Toeplitz matrix, and let  $v \in \mathbb{C}^n$  be arbitrary. It is well known that  $Cv$  can be computed on  $O(n \log n)$  arithmetic operations using FFTs. Hence  $H_0 v, (R^H R)^{-1} v$ , and  $(R^H R)v$  can be computed in  $O(n \log n)$  arithmetic operations, where we use (1.18)–(1.19). Each iteration of the Lanczos algorithm given in [9, p. 324] therefore requires only  $O(n \log n)$  arithmetic operations.

The computation of singular values of  $H$  is important in Hankel norm approximation problems of systems theory, such as the model reduction problem [4]. The approximation of functions by the Carathéodory-Fejér method yields another application [5, 12].

Other methods for reducing the singular value problem for  $H$  to a finite dimensional one have been described by Kung and Gutknecht [5] and Young [13]. These methods, however, do not preserve symmetry. Moreover, Young's

approach requires generally  $O(n^3)$  arithmetic operations to compute the matrices required.

REMARK 1.1. Formulas (1.3)–(1.8) and the proof of Proposition 1.1 require distinct  $\lambda_k$ . This restriction can be removed. Assume first that  $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ . Then (1.3)–(1.4) have to be replaced by

$$\rho(\lambda) = \sum_{k=1}^n \frac{\alpha_k}{(\lambda - \lambda_1)^k}, \quad (1.3')$$

$$\eta_j = \sum_{k=1}^n \frac{\alpha_k}{\lambda^k} \left[ \sum_{j=0}^{\infty} \left( \frac{\lambda_1}{\lambda} \right)^j \right]^k. \quad (1.4')$$

In (1.5)  $A$  has to be replaced by the upper triangular Hankel matrix

$$A = [\alpha_{j+k+1}]_{j,k=0}^{n-1} \in \mathbb{C}^{n \times n}, \quad \alpha_p := 0, \quad p > n.$$

The matrix  $\Lambda$  in (1.6) has to be replaced by the Jordan matrix with all diagonal elements equal to  $\lambda_1$  and all superdiagonal elements equal to one. The matrix  $V_0$  in (1.7) must be replaced by the confluent Vandermonde matrix. For instance, we obtain for  $n = 3$

$$\Lambda = \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 1 & & \\ \lambda_1 & 1 & \\ \lambda_1^2 & 2\lambda_1 & 1 \end{bmatrix}.$$

With  $A$ ,  $\Lambda$ , and  $V_0$  modified as described, we define  $V_j$  and  $V$  by (1.8)–(1.9),  $U$  by (1.11), and  $H_0$  by (1.12). Then (1.10) and (1.13) hold and  $H_0$  is the leading principal  $n \times n$  submatrix of  $H$ . Also (1.14)–(1.19) remain valid. Proposition 1.1 can be shown by replacing (1.4) with (1.4'), and by bounding the sum

$$\left\| \sum_{j=0}^{\infty} (\Lambda^H \Lambda)^{nj} \right\|_2^2$$

where  $\Lambda$  now is a Jordan matrix. This sum is bounded if  $|\lambda_1| < 1$ , and the proposition remains valid.

In general, when the  $\lambda_k$  are of arbitrary multiplicity,  $A$  in (1.5) has to be replaced by a block diagonal matrix, where each block is an upper triangular Hankel matrix. The blocks are of the same sizes as the multiplicities of the  $\lambda_k$ , and the number of blocks equals the number of distinct  $\lambda_k$ .  $\Lambda$  in (1.6) is replaced by a Jordan matrix with Jordan boxes of the same sizes as the multiplicities of the  $\lambda_k$ , and the number of boxes equal to the number of distinct  $\lambda_k$ .  $V_0$  in (1.7) is replaced by an appropriate confluent Vandermonde matrix. With these changes (1.10)–(1.19) are valid, and so is Proposition 1.1. We omit the details, since the numerical computations are independent of the multiplicity of the  $\lambda_k$ .

## 2. THE SYMMETRIC SINGULAR VALUE DECOMPOSITION

In this section we present an elementary proof of Takagi's theorem, i.e., we show the existence of a symmetric singular value decomposition of a complex symmetric matrix. Let  $C = C^T \in \mathbb{C}^{n \times n}$ , and define  $A, B \in \mathbb{R}^{n \times n}$  by  $C := A + iB$ ,  $i := \sqrt{-1}$ . Then  $A = A^T$  and  $B = B^T$ , so the matrix

$$\tilde{C} := \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

is real and symmetric. Let  $\{\sigma_j\}_{j=1}^r$  be the positive eigenvalues of  $\tilde{C}$  and form

$$\Sigma := \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_r].$$

Let

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Sigma \quad (2.1)$$

with

$$U, V \in \mathbb{R}^{n \times r}$$

and

$$U^T U + V^T V = I_r.$$

Write (2.1) as

$$\begin{aligned} AU + BV &= U\Sigma, \\ BU - AV &= V\Sigma, \end{aligned} \quad (2.2)$$

and note that (2.2) also can be written as

$$\begin{aligned} AV + B(-U) &= V(-\Sigma), \\ BV - A(-U) &= (-U)(-\Sigma), \end{aligned}$$

i.e.,

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} V \\ -U \end{bmatrix} = \begin{bmatrix} V \\ -U \end{bmatrix} (-\Sigma) \quad (2.3)$$

with

$$V^T V + (-U)^T (-U) = I_r.$$

Hence  $\tilde{C}$  has at least  $r$  negative eigenvalues. We could also have let  $\sigma_j$  be the negative eigenvalues of  $\tilde{C}$ , and then (2.3) would have given us positive ones. We therefore may assume that  $\pm\sigma_1, \pm\sigma_2, \dots, \pm\sigma_r$  are all the nonzero eigenvalues of  $\tilde{C}$ .

Since eigenvectors associated with distinct eigenvalues of a real symmetric matrix are orthogonal, we have

$$0 = [V^T, -U^T] \begin{bmatrix} U \\ V \end{bmatrix} = V^T U - U^T V.$$

The spectral resolution of  $\tilde{C}$  is thus

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} = \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} \Sigma & \\ & -\Sigma \end{bmatrix} \begin{bmatrix} U^T & V^T \\ V^T & -U^T \end{bmatrix},$$

which yields

$$\begin{aligned} A &= U\Sigma U^T - V\Sigma V^T, \\ B &= V\Sigma U^T + U\Sigma V^T. \end{aligned}$$

Therefore

$$\begin{aligned} C &= A + iB = U\Sigma U^T - V\Sigma V^T + i(V\Sigma U^T + U\Sigma V^T) \\ &= (U + iV)\Sigma(U^T + iV^T) = W\Sigma W^T = \sum_{k=1}^r \sigma_k w w_k^T, \end{aligned}$$



where

$$U + iV =: W = [w_1, w_2, \dots, w_r], \quad w_k \in \mathbb{C}^n.$$

Moreover

$$W^H W = (U^T - iV^T)(U + iV) = (U^T U + V^T V) + i(U^T V - V^T U) = I_r.$$

If  $r < n$  then one may replace  $\Sigma$  by

$$\Sigma_0 := \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_r, 0, \dots, 0] \in \mathbb{R}^{n \times n}$$

and  $W$  by

$$W_0 := [w_1, w_2, \dots, w_r, w_{r+1}, \dots, w_n] \in \mathbb{C}^{n \times n},$$

where  $w_{r+1}, \dots, w_n \in \mathbb{C}^n$  are chosen so that  $W_0^H W_0 = I_n$ .

### 3. A SIMPLE EXPRESSION FOR $(R^H R)^{-1}$

In this section we derive (1.18). Introduce the Frobenius matrix

$$F := [e_2, e_3, \dots, e_n, -f] \in \mathbb{C}^{n \times n},$$

where

$$\begin{aligned} e_j &:= [\delta_{1j}, \delta_{2j}, \dots, \delta_{nj}]^T \in \mathbb{R}^n, \quad 2 \leq j \leq n, \\ f &:= [\chi_0, \chi_1, \dots, \chi_{n-1}]^T \in \mathbb{C}^n. \end{aligned} \tag{3.1}$$

Then  $F$  is the companion matrix of  $\chi$ , and

$$F^T V_0 = V_0 \Lambda. \tag{3.2}$$

Throughout this section  $V_0$  and  $\Lambda$  are defined by (1.6)–(1.7) if the  $\lambda_k$  are

distinct. For confluent  $\lambda_k$  we modify  $V_0$  and  $\Lambda$  according to Remark 1.1. The following lemma shows that

$$G := \overline{R^H R} \quad (3.3)$$

satisfies a Stein equation. This will enable us to obtain a simple expression for  $G^{-1}$  by an application of the Sherman-Morrison-Woodbury formula.

LEMMA 3.1.  *$G$  is the unique solution of the Stein equation*

$$X - F^n X F^{nH} = I_n, \quad X \in \mathbb{C}^{n \times n}. \quad (3.4)$$

*Proof.* By (1.8), (1.9), and (1.11) we obtain

$$R^H R = U^H U = \sum_{k=0}^{\infty} V_0^{-H} (\Lambda^{nk})^H V_0^H V_0 \Lambda^{nk} V_0^{-1}, \quad (3.5)$$

and (3.2) yields now

$$G = \sum_{k=0}^{\infty} F^{nk} (F^{nk})^H. \quad (3.6)$$

The series in (3.5)–(3.6) converge because  $|\lambda_k| < 1$  for all  $k$ . Substitution of (3.6) into (3.4) shows that  $G$  solves (3.4). The uniqueness follows from  $|\lambda_k| < 1$  for all  $k$ . The latter can be seen by a similarity transform of  $F^n$  to Schur triangular form. ■

Introduce the cyclic downshift operator in  $\mathbb{C}^{2n}$ :

$$E := [e_2, e_3, \dots, e_{2n}, e_1] \in \mathbb{C}^{2n \times 2n},$$

where

$$e_j := [\delta_{1j}, \delta_{2j}, \dots, \delta_{2n,j}]^T \in \mathbb{R}^{2n}. \quad (3.7)$$

Let

$$t := [\chi_0, \chi_1, \dots, \chi_n, 0, 0, \dots, 0]^T \in \mathbb{C}^{2n},$$

and define the Toeplitz matrix  $T$  of parallelogram form:

$$T := [t, Et, E^2t, \dots, E^{n-1}t] \in \mathbb{C}^{2n \times n}. \quad (3.8)$$

Let  $T_0$  be the leading  $n \times n$  submatrix of  $T$ , and let  $T_1$  be the trailing  $n \times n$  submatrix of  $T$ . Then  $T_0$  is a left triangular Toeplitz matrix, and  $T_1$  is a unit right triangular Toeplitz matrix.

EXAMPLE 3.1. Let  $n = 3$ . Then

$$T = \begin{bmatrix} \chi_0 & & & \\ \chi_1 & \chi_0 & & \\ \chi_2 & \chi_2 & \chi_0 & \\ \chi_3 & \chi_2 & \chi_1 & \\ & \chi_3 & \chi_2 & \\ & & \chi_3 & \end{bmatrix}, \quad T_0 = \begin{bmatrix} \chi_0 & & \\ \chi_1 & \chi_0 & \\ \chi_2 & \chi_1 & \chi_0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} \chi_3 & \chi_2 & \chi_1 \\ & \chi_3 & \chi_2 \\ & & \chi_3 \end{bmatrix},$$

where we note that  $\chi_3 = 1$ .

LEMMA 3.2. Let  $T_0$  and  $T_1$  be defined as above. Then

$$T_0^H T_0 + T_1^H T_1 = T_0 T_0^H + T_1 T_1^H. \quad (3.9)$$

*Proof.* Let  $N := T^H T = T_0^H T_0 + T_1^H T_1$ . We first show that  $N$  is a Toeplitz matrix. Let  $e_j$  be defined by (3.1). Then by (3.8) we have for  $1 \leq j, k \leq n$

$$e_j^T N e_k = e_j^T T^H T e_k = t^H (E^H)^{j-1} E^{k-1} t = t^H E^{k-j} t,$$

where we have used that  $E^H = E^{-1}$ . We next define the reversal matrix

$$J := [e_n, e_{n-1}, \dots, e_1] \in \mathbb{R}^{n \times n}.$$

Toeplitz matrices are countersymmetric, i.e.  $N = JN^T J$ . Using that  $N$  is countersymmetric and Hermitian yields

$$\begin{aligned} T_0^H T_0 + T_1^H T_1 &= N = JN^T J = J\bar{N}J = J(T_0^T \bar{T}_0 + T_1^T \bar{T}_1)J \\ &= JT_0^T J \cdot J\bar{T}_0 J + JT_1^T J \cdot J\bar{T}_1 J = T_0 T_0^H + T_1 T_1^H. \end{aligned} \quad \blacksquare$$

The next lemma presents a Gaussian factorization of  $F^n$  in terms of  $T_0$  and  $T_1$ . This will be used together with Lemma 3.1 to express  $G^{-1}$  in terms of  $T_0$  and  $T_1$ .

LEMMA 3.3.

$$F^n = -T_0 T_1^{-1}. \quad (3.10)$$

*Proof.* We first show that

$$[T_0^T, T_1^T] \begin{bmatrix} V_0 \\ V_0 \Lambda^n \end{bmatrix} = 0. \quad (3.11)$$

Let  $e_j$  be defined by (3.7), and assume for the moment that the  $\lambda_k$  are distinct. Then

$$e_j^T [T_0^T, T_1^T] \begin{bmatrix} V_0 \\ V_0 \Lambda^n \end{bmatrix} e_k = \chi(\lambda_k) \lambda_k^{j-1}, \quad (3.12)$$

and the right hand side vanishes for  $1 \leq j, k \leq n$ . If the  $\lambda_k$  are confluent, then the right hand side of (3.12) contains derivatives of  $\chi(\lambda)$  evaluated at  $\lambda_k$ . The right hand side of (3.12), however, still vanishes and (3.11) holds.

We now write (3.11) as

$$T_0^T V_0 + T_1^T V_0 \Lambda^n = 0$$

and apply (3.2). This shows (3.10). ■

We are now in a position to show (1.18). By (3.4)  $G$  satisfies

$$G = I + F^n G F^{nH},$$

and an application of the Sherman-Morrison-Woodbury formula yields

$$G^{-1} = (I + F^n G F^{nH})^{-1} = I - F^n (G^{-1} + F^{nH} F^n)^{-1} F^{nH}. \quad (3.13)$$

We now determine an expression for

$$Y := I - G^{-1}. \quad (3.14)$$

Substitute  $Y$  and (3.10) into (3.13) to obtain

$$Y = T_0(T_0^H T_0 + T_1^H T_1 - T_1^H Y T_1)^{-1} T_0^H. \quad (3.15)$$

In order to determine a simple expression for  $Y$  from (3.15) we need the following observation, which is also central to section 4.  $T_0$  and  $T_1^{-H}$  are both left triangular  $n \times n$  Toeplitz matrices. Multiplication of  $T_0$  with  $T_1^{-H}$  can be identified with polynomial multiplication; see [6, Section 1.3] and Section 4. Since multiplication of polynomials commutes, we obtain

$$T_0 T_1^{-H} = T_1^{-H} T_0. \quad (3.16)$$

From the correspondence between polynomials and left triangular Toeplitz matrices it also follows that  $T_0 T_1^{-H}$  is a left triangular Toeplitz matrix.

LEMMA 3.4. *Equation (3.15) has the unique solution*

$$Y = T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 T_1^{-H} T_1^{-1} T_0^H. \quad (3.17)$$

*Proof.* Uniqueness follows from (3.14) and the fact that (3.4) has a unique solution. From (3.16) we obtain

$$T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 T_1^{-H} T_1^{-1} T_0^H. \quad (3.18)$$

Now substitute

$$Y = T_1^{-H} T_0 T_0^H T_1^{-1}$$

into (3.15). We obtain

$$T_1^{-H} T_0 T_0^H T_1^{-1} = T_0(T_0^H T_0 + T_1^H T_1 - T_0 T_0^H)^{-1} T_0^H. \quad (3.19)$$

An application of (3.9) reduces (3.19) to (3.18). The latter has already been shown to be valid. Therefore (3.17) solves (3.15). ■

Let

$$B_0 := \bar{T}_0 T_1^{-T} = T_1^{-T} \bar{T}_0. \quad (3.20)$$

Then  $B_0$  is a left triangular  $n \times n$  Toeplitz matrix. By (3.14) and (3.17)

$$G^{-1} = I - \bar{B}_0 B_0^T = I - B_0^T \bar{B}_0.$$

From (3.3) it now follows that

$$(R^H R)^{-1} = I - B_0 B_0^H. \quad (3.21)$$

#### 4. COMPUTATION OF $H_0$ AND $B_0$

We summarize some results in [6, Section 1.3] and [7, Section 13.9] in order to show that the elements of  $H_0$  and  $B_0$  can be computed in  $O(n \log n)$  arithmetic operations from the coefficients  $\chi_j$  of  $\chi$  and  $\pi_j$  of  $\pi$ ; see (1.1). To a polynomial or power series

$$\zeta(\lambda) := \sum_{j=0}^{n-1} \zeta_j \lambda^j + O(\lambda^n)$$

we associate the left triangular  $n \times n$  Toeplitz matrix

$$Z = [\zeta_{j-k}]_{j,k=0}^{n-1}, \quad \zeta_j = 0 \quad \text{for } j < 0,$$

and we write  $\zeta \rightarrow Z$ . If  $\xi(\lambda)$  is a polynomial and  $X$  a left triangular  $n \times n$  Toeplitz matrix such that  $\xi \rightarrow X$ , then it is easily seen that  $\zeta \xi \rightarrow ZX$ . In particular,  $ZX$  is a left triangular  $n \times n$  Toeplitz matrix. From  $\xi \zeta = \zeta \xi$  and  $\xi \zeta \rightarrow XZ$  it follows that  $ZX = XZ$ .

Assume that  $\zeta_0 \neq 0$ , and let  $1/\zeta \rightarrow Z'$ . Then  $(1/\zeta) \cdot \zeta \rightarrow I$ ,  $Z'Z$ , and  $ZZ'$ . We obtain  $Z' = Z^{-1}$ , and therefore  $Z^{-1}$  is a left triangular Toeplitz matrix.

**EXAMPLE 4.1.** We have  $\chi \rightarrow T_0$ . Let

$$\tilde{\chi}(\lambda) := \lambda^n \bar{\chi} \frac{1}{\lambda} = \sum_{j=0}^n \bar{\chi}_{n-j} \lambda^j. \quad (4.1)$$

Then  $\tilde{\chi} \rightarrow T_1^H$ , and the Blaschke product

$$\frac{\chi}{\tilde{\chi}} \rightarrow T_0 T_1^{-H} = \bar{B}_0. \quad (4.2)$$

Now let  $\xi(\lambda)$  and  $\zeta(\lambda)$  be arbitrary polynomials such that  $\zeta(0) \neq 0$ . Henrici [7, Theorem 13.9e] shows that the first  $n$  coefficients in the MacLaurin expansion of  $\xi(\lambda)/\zeta(\lambda)$  can be computed in  $O(n \log n)$  multiplications. The proof uses FFT. It is easily seen that the number of additions also is  $O(n \log n)$ .

From  $\chi_n = 1$  and (4.1) we obtain  $\tilde{\chi}(0) \neq 0$ . Hence, the first  $n$  terms in the MacLaurin expansion of  $\chi/\tilde{\chi}$  can be computed in  $O(n \log n)$  arithmetic operations. By (4.2) therefore  $T_0 T_1^{-H} = B_0$  can be computed in  $O(n \log n)$  arithmetic operations.

Because  $\lambda^n \chi(1/\lambda) \neq 0$  for  $\lambda = 0$ , we can compute the first  $n$  terms in the MacLaurin expansion of

$$\frac{\lambda^n \pi(1/\lambda)}{\lambda^n \chi(1/\lambda)} = \sum_{j=0}^{n-1} \eta_j \lambda^{j+1} + O(\lambda^n)$$

in  $O(n \log n)$  arithmetic operations. This shows that  $H_0$  can be computed in  $O(n \log n)$  arithmetic operations.

## 5. A FACTORIZATION OF $R^H R$

It follows from (3.3) and (3.20)–(3.21) that

$$G^{-1} = \overline{(R^H R)}^{-1} = I - \overline{B_0 B_0^H} = I - T_1^{-H} T_0 T_0^H T_1^{-1}, \quad (5.1)$$

and therefore

$$T_1^H G^{-1} T_1 = T_1^H T_1 - T_0 T_0^H =: M_0^{-1}. \quad (5.2)$$

The expression defining  $M_0^{-1}$  is a Gohberg-Semencul formula for the inverse of an  $n \times n$  Toeplitz matrix; see, e.g., [8, p. 152]. We denote this Toeplitz matrix by  $M_0$ . From the left hand expression (5.2) and the nonsingularity of

$T_1$  and  $R$  it follows that  $M_0$  is Hermitian and positive definite. The desired factorization of  $R^H R$  is

$$R^H R = \overline{T_1 M_0 T_1^H}.$$

We will now show how  $M_0$  can be computed. The computation involves running the Levinson algorithm backwards.

Consider the related Gohberg-Semencul formula (see, e.g., [8, Theorem 18.1, p. 148] of [1])

$$M_1^{-1} = \begin{bmatrix} \chi_n & \chi_{n-1} & \cdots & \chi_0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & \chi_{n-1} \\ & & & \chi_n \end{bmatrix}^H \begin{bmatrix} \chi_n & \chi_{n-1} & \cdots & \chi_0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & \chi_{n-1} \\ & & & \chi_n \end{bmatrix} - \begin{bmatrix} 0 & & & & \\ \chi_0 & \ddots & & & \\ \chi_1 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \chi_{n-1} & \cdots & \chi_1 & \chi_0 & 0 \end{bmatrix} \begin{bmatrix} 0 & & & & \\ \chi_0 & \ddots & & & \\ \chi_1 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \chi_{n-1} & \cdots & \chi_1 & \chi_0 & 0 \end{bmatrix}^H, \quad (5.3)$$

where the four triangular Toeplitz matrices define the inverse of an  $(n+1) \times (n+1)$  Hermitian Toeplitz matrix. Denote this Toeplitz matrix by  $M_1$ . Then  $M_0$  is the leading principal  $n \times n$  submatrix of  $M_1$ ; see [8, Theorems 18.1–18.2].

Let  $R_1 := [\rho_{jk}]_{j,k=0}^n \in \mathbb{C}^{(n+1) \times (n+1)}$  be the unit right triangular matrix, and let  $D_1 := \text{diag}[\delta_0, \delta_1, \dots, \delta_n]$  be the diagonal matrix such that

$$R_1^H M_1 R_1 = D_1. \quad (5.4)$$

Given  $M_1 = [\mu_{j-k}]_{j,k=0}^n$ , the matrices  $R_1$  and  $D_1$  can be computed by the Levinson algorithm, and by comparing  $R_1$  with (5.3) one finds that

$$\rho_{jn} = \chi_j, \quad 0 \leq j \leq n, \quad \text{and} \quad \delta_n = \chi_n;$$

see, e.g., [1]. We now apply the recursion formula in Levinson's algorithm



backwards in order to determine  $R_1$  and  $D_1$  from the last column of  $R_1$  and  $\delta_n$ . Then the recursion formula is used forwards to determine  $M_0$ . We will also obtain a Hermitian factorization of  $R^H R$  into triangular matrices.

#### BACKWARD LEVINSON ALGORITHM.

input:  $[\rho_{jn}]_{j=0}^n, \delta_n$ ; output:  $R_1, D_1$ , Schur parameters  $\{\gamma_j\}_{j=1}^n$  of  $M_0$ ;  
**for**  $k := n, n-1, n-2, \dots, 1$  **do**  
     $\gamma_k := \rho_{0k}; \rho_{k-1, k-1} := 1$ ;  
    **for**  $j := 1, 2, \dots, \text{integer part}(k/2)$  **do**  
        solve for  $\rho_{j-1, k-1}$  and  $\rho_{k-1-j, k-1}$  the linear system of equations  

$$\begin{bmatrix} 1 & \gamma_k \\ \bar{\gamma}_k & 1 \end{bmatrix} \begin{bmatrix} \rho_{j-1, k-1} \\ \bar{\rho}_{k-1-j, k-1} \end{bmatrix} = \begin{bmatrix} \rho_{j, k} \\ \bar{\rho}_{k-j, k} \end{bmatrix};$$
  
     $\delta_{k-1} := [\delta_k / (1 - |\gamma_k|)] / (1 + |\gamma_k|)$ ;

LEVINSON RECURSION FOR COMPUTING  $M_0 = [\mu_{j-k}]_{j, k=0}^{n-1}$ .

input:  $R_1, D_1, \{\gamma_j\}_{j=1}^n$ ; output:  $\{\mu_j\}_{j=0}^{n-1}$ ;  
 $\mu_0 := \delta_0; \mu_1 := -\delta_0 \bar{\gamma}_1$ ;  
**for**  $k := 1, 2, \dots, n-1$  **do**  
     $\mu_{k+1} := -\delta_k \bar{\gamma}_{k+1} - \sum_{j=1}^k \mu_j \bar{\rho}_{j-1, k}$ ;

Hence  $M_0$ ,  $R_1$ , and  $D_1$  are computed in  $O(n^2)$  arithmetic operations from the coefficients of  $\chi$ . Let  $R_0$  and  $D_0$  denote the  $n \times n$  leading principal submatrices of  $R_1$  and  $D_1$  respectively. Similarly to (5.4) we have

$$R_0^H M_0 R_0 = D_0. \quad (5.5)$$

Because  $M_0$  is positive definite, so is  $D_0$ . Therefore  $D_0^{1/2}$  can easily be computed. We obtain from (5.1)–(5.2) and (5.5), with  $\hat{R} := D_0^{1/2} R_0^{-1}$ ,

$$R^H R = (\hat{R} T_1^H)^T \overline{\hat{R} T_1^H}. \quad (5.6)$$

The right hand side of (5.6) is a Hermitian factorization into triangular matrices. It can be computed in  $O(n^2)$  arithmetic operations from the coefficients of  $\chi$ .

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